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Characterization of Hilbert Spaces by the Strong Law of Large Numbers

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Hilbert spaces are characterized through the validity of the strong law of large numbers. Other characterizations of Hilbert spaces are also given at the same time in this note. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let X be a real separable Banach space and X^* its topological dual space. Let $(\xi_n)_{n\geqslant 1}$ be a sequence of X-valued random variables defined on a basic probability space $(\Omega, \mathscr{A}, P)$ satisfying

$$E\xi_n = 0 \qquad \text{for all } n \geqslant 1, \tag{1.1}$$

where E denotes the expectation (that is, Bochner integral with respect to P). Then (ξ_n) is said to *satisfy the strong law of large numbers* if

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \to 0 \qquad \text{a. s.} \quad (=\text{almost surely}). \tag{1.2}$$

We shall then study under what conditions the strong law of large numbers holds. In [1] Beck has considered the condition

$$\sup_{n\geqslant 1} E\|\xi_n\|^2 < \infty \tag{1.3}$$

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and in [5] Hoffmann–Jørgensen and Pisier have considered the condition

$$\sum_{n=1}^{\infty} n^{-2} E\|\xi_n\|^2 < \infty. \tag{1.4}$$

In what follows, we denote by $L^2(X)$ the Banach space of all X-valued random variables which satisfy $E\|\xi\|^2 < \infty$ and, for each ξ in $L^2(X)$ denote by S_ξ its *covariance operator* (see [7]); that is, the nuclear operator of X^* into X which is defined by

$$S_\xi f = E\langle \xi, f\rangle\, \xi - \langle E\xi, f\rangle\, E\xi, \qquad f \in X^*.$$

Further we denote by $\mathcal{N}(X^*, X)$ the Banach space of all nuclear operators of X^* into X with the nuclear norm $\nu(\cdot)$ (see [4]).

In this note we shall consider a condition

$$\sum_{n=1}^{\infty} n^{-2} S_{\xi_n} \qquad \text{converges in } \mathcal{N}(X^*, X), \tag{1.5}$$

which is weaker than the above two conditions (1.3) and (1.4), and show that the strong law of large numbers holds for any sequence $(\xi_n)_{n\geqslant 1}$ of independent X-valued random variables satisfying (1.1) and (1.5) if and only if X is isomorphic to a Hilbert space. Other characterizations of Hilbert spaces are also given at the same time in the following theorem. Throughout this note, the letter R will denote the set of real numbers.

THEOREM. *For a real separable Banach space X, the following assertions* (1)–(4) *are equivalent*:

(1) *X is isomorphic to a Hilbert space.*

(2) *For any sequence $(\xi_n)_{n\geqslant 1} \subset L^2(X)$ satisfying* (1.1), *ξ_n converges in $L^2(X)$ if* (*and only if*) *the following two conditions are satisfied*:

(a) *For each $f \in X^*$, $\langle \xi_n, f\rangle$ converges in $L^2(R)$.*

(b) *The set $\{S_{\xi_n}\}$ is relatively compact in $\mathcal{N}(X^*, X)$.*

(3) *For any sequence $(\xi_n)_{n\geqslant 1}$ of independent X-valued random variables satisfying* (1.1), *$\sum_{n=1}^{\infty} \xi_n$ converges a. s. if*

$$\sum_{n=1}^{\infty} S_{\xi_n} \qquad \textit{converges in } \mathcal{N}(X^*, X). \tag{1.6}$$

(4) *The strong law of large numbers holds for any sequence $(\xi_n)_{n\geqslant 1}$ of independent X-valued random variables satisfying* (1.1) *and* (1.5).

2. Auxiliary Results

Before starting to prove the theorem, in this section we shall characterize relative compactness of sets and L^2-convergence of sequences in $L^2(X)$ when X is a Hilbert space.

Denote by $l^p(X)$, $1 \leqslant p < \infty$, the Banach space of all sequences $x = (x_n)_{n \geqslant 1}$ of vectors in a Banach space X which satisfy $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. We shall use the following lemma to prove results in this section.

Lemma 1 [2]. *A set $K \subset l^p(X)$ is relatively compact if and only if:*

(i) *For each $n \geqslant 1$, the set $K(n) = \{x_n;\ x \in K\}$ is relatively compact in X.*

(ii) $\lim_{N \to \infty} \sup_{x \in K} \sum_{n=N}^{\infty} \|x_n\|^p = 0$.

Let H be a real separable Hilbert space with the inner product $(\cdot, \cdot)$. Then $\mathcal{N}(X^*, X)$ coincides with the Banach space $\mathcal{T}(H)$ of all trace class operators on H.

Proposition 2. *A set $K \subset L^2(H)$ is relatively compact if and only if the following conditions are satisfied:*

(1) *For each $x \in H$, the set $\{(\xi, x);\ \xi \in K\}$ is relatively compact in $L^2(R)$.*

(2) *The set $\{E\xi;\ \xi \in K\}$ is relatively compact in H and the set $\{S_\xi;\ \xi \in K\}$ is relatively compact in $\mathcal{T}(H)$.*

Proof. Assume first that K is relatively compact. Then condition (1) is satisfied since the map $\xi \to (\xi, x)$ of $L^2(H)$ into $L^2(R)$ is continuous. Condition (2) follows from an easily verified inequality,

$$\nu(S_\xi - S_\eta) \leqslant (E\|\xi - \eta\|^2)^{1/2} \cdot \{(E\|\xi\|^2)^{1/2} + (E\|\eta\|^2)^{1/2}\}$$

for all ξ, η in $L^2(H)$ satisfying $E\xi = E\eta = 0$.

Conversely, assume conditions (1) and (2). Let $\{\varphi_n\}$ be a basis of H and let us consider the set $M \equiv \{((\xi, \varphi_n))_{n \geqslant 1};\ \xi \in K\}$ of elements in $l^2(L^2(R))$. Then M is relatively compact in $l^2(L^2(R))$, since it satisfies conditions (i) and (ii) of Lemma 1. In fact, condition (i) of Lemma 1 is satisfied from hypothesis (1). Since the map $x \to ((x, \varphi_n))_{n \geqslant 1}$ of H into $l^2(R)$ and the map $S \to ((S\varphi_n, \varphi_n))_{n \geqslant 1}$ of $\mathcal{T}(H)$ into $l^1(R)$ are continuous, it follows from hypothesis (2) that the set $\{((E\xi, \varphi_n))_{n \geqslant 1};\ \xi \in K\}$ and the set $\{((S_\xi\varphi_n, \varphi_n))_{n \geqslant 1};\ \xi \in K\}$ are relatively compact in $l^2(R)$ and $l^1(R)$, respectively.

Thus we have

$$\lim_{N\to\infty} \sup_{\xi\in K} \sum_{n=N}^{\infty} E|(\xi, \varphi_n)|^2$$

$$\leqslant 2\cdot\left\{\lim_{N\to\infty} \sup_{\xi\in K} \sum_{n=N}^{\infty} |(E\xi, \varphi_n)|^2 + \lim_{N\to\infty} \sup_{\xi\in K} \sum_{n=N}^{\infty} (S\varphi_n, \varphi_n)\right\} = 0;$$

therefore condition (ii) of Lemma 1 is also satisfied. We define now a continuous map of $l^2(L^2(R))$ into $L^2(H)$ as follows:

$$(\theta x)(\cdot) = \sum_{n=1}^{\infty} x_n(\cdot)\,\varphi_n, \qquad x = (x_n(\cdot))_{n\geqslant 1} \in l^2(L^2(R)),$$

where the series is norm convergent in $L^2(H)$. Then we have $\theta M = K$ and consequently K itself is relatively compact in $L^2(H)$. From Proposition 2, using the standard argument, we have

COROLLARY 3. *A sequence $(\xi_n)_{n\geqslant 1} \subset L^2(H)$ converges in $L^2(H)$ if and only if the following conditions are satisfied*:

(1) *For each $x\in H$, (ξ_n, x) converges in $L^2(R)$.*

(2) *The set $\{E\xi_n\}$ is relatively compact in H and the set $\{S_{\xi_n}\}$ is relatively compact in $\mathscr{T}(H)$.*

3. PROOF OF THEOREM

$(1)\Rightarrow(2)$ follows from Corollary 3. $(2)\Rightarrow(3)$: Set $\zeta_n = \sum_{k=1}^n \xi_k$. It is sufficient to show that the sequence $(\zeta_n)_{n\geqslant 1}$ satisfies conditions (a) and (b) of (2) because $(\zeta_n)_{n\geqslant 1}$ converges a.s. if it converges in $L^2(X)$ (see [5, Lemma 1.2]). Condition (b) follows from (1.6) since $\sum_{k=1}^n S_{\xi_k} = S_{\zeta_n}$ for all $n\geqslant 1$. On the other hand, for each $f\in X^*$ we have

$$E\left|\sum_{k=m}^{n} \langle \xi_k, f\rangle\right|^2 = \sum_{k=m}^{n} \langle S_{\xi_k} f, f\rangle \leqslant \nu\left(\sum_{k=m}^{n} S_{\xi_k}\right)\cdot \|f\|^2$$

for all $n > m \geqslant 1$, and this, together with (1.6), implies condition (a). $(3)\Rightarrow(4)$: By (1.1), (1.5), and assumption (3) we have that $\sum_{n=1}^{\infty} \xi_n/n$ converges a.s. since $\sum_{k=1}^n k^{-2} S_{\xi_k} = \sum_{k=1}^n S_{\xi_k/k}$ for all $n\geqslant 1$. Now we notice that Kronecker's lemma is valid in any Banach space (with the same proof as in the real case), so we have that $\sum_{k=1}^n \xi_k/n$ converges to 0 as $n\to\infty$. In order to prove $(4)\Rightarrow(1)$ we need definitions of "type" and "cotype" and a result of [6].

Let $(\gamma_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables. A Banach space X is said to be *of type* 2 if for every sequence $(x_n)_{n\geqslant 1}$ in X such that $\sum_{n=1}^{\infty}\|x_n\|^2<\infty$, we have that the series $\sum_{n=1}^{\infty}\gamma_n x_n$ converges a.s. and is said to be *of cotype* 2 if for every sequence $(x_n)_{n\geqslant 1}$ such that the series $\sum_{n=1}^{\infty}\gamma_n x_n$ converges a.s., we have $\sum_{n=1}^{\infty}\|x_n\|^2<\infty$. From a standard argument using the closed graph theorem we find that a Banach space X is of type 2 and of cotype 2 if and only if there exists a constant $C\geqslant 1$ such that for any positive integer n and any $x_1, x_2,..., x_n$ in X,

$$C^{-1}\sum_{k=1}^{n}\|x_k\|^2\leqslant E\left\|\sum_{k=1}^{n}\gamma_k x_k\right\|^2\leqslant C\sum_{k=1}^{n}\|x_k\|^2.$$

Then Kwapién has shown in [6, Proposition 3.1] that if X is of type 2 and of cotype 2 then it is isomorphic to a Hilbert space.

Now suppose that (4) holds, and we prove that if X is of type 2 and of cotype 2. First, since condition (1.5) is weaker than (1.4) it follows from Theorem 2.1 in [5] that X is of type 2. Next we prove that X is of cotype 2. Suppose to the contrary that there exists a sequence $(x_i)_{i\geqslant 1}$ such that $\sum_{i=1}^{\infty}\gamma_i x_i$ converges a.s., but $\sum_{i=1}^{\infty}\|x_i\|^2=\infty$. (Without loss of generality we may assume that $x_i\neq 0$ for all $i\geqslant 1$.) If we set $a_k=\sum_{i=1}^{k}\|x_i\|^2$ then $a_k\to\infty$ and also $\sum_{k=1}^{\infty}\|x_k\|^2/a_k=\infty$. Define a sequence $(\xi_k)_{k\geqslant 1}$ of independent X-valued random variables satisfying (1.1) with distributions such that

$$\begin{gathered}\Pr\left(\xi_k=-\frac{ka_k^{1/2}x_k}{\|x_k\|}\right)=\Pr\left(\xi_k=\frac{ka_k^{1/2}x_k}{\|x_k\|}\right)=\frac{1}{2}\cdot\frac{\|x_k\|^2}{a_k}\\ \Pr(\xi_k=0)=1-\|x_k\|^2/a_k.\end{gathered}\tag{3.1}$$

Then by the Borel–Cantelli lemma, ξ_n/n does not converge to 0 a.s. so that

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \qquad \text{does not converge to 0 a.s.}\tag{3.2}$$

Now let us consider a sequence $(\eta_n)_{n\geqslant 1}$ of X-valued Gaussian random variables defined by $\eta_n=\sum_{k=1}^{n}\gamma_k x_k$. Then, by (3.1) it is easy to show that $S_{\eta_n}=\sum_{k=1}^{n}k^{-2}S_{\xi_k}$ for all $n\geqslant 1$. Since η_n converges a.s., by Theorem 1 [3] $S_{\eta_n}=\sum_{k=1}^{n}k^{-2}S_{\xi_k}$ converges in $\mathcal{N}(X^*, X)$. Therefore from assumption (4) it follows that $\sum_{k=1}^{n}\xi_k/n$ converges to 0 a.s. and this contradicts (3.2). The proof is now complete.

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